MAGIC STARS

MARIÀN TRENKLER*

You have probably already met with the following brain-twisters whose author is the English mathematician Henry Dudeney, [1].

Problem 1. Into the circlets of the star $\mathbb{S}_5$ in Figure 1 write ten different numbers from the set $\{1, 2, 3, \ldots, 12\}$ in such a way that the four numbers on each line sum to twenty-four.

Problem 2. Into the circlets of the star $\mathbb{T}_8$ in Figure 1 write numbers $1, 2, \ldots, 16$ so that the four numbers on each line sum to thirty.

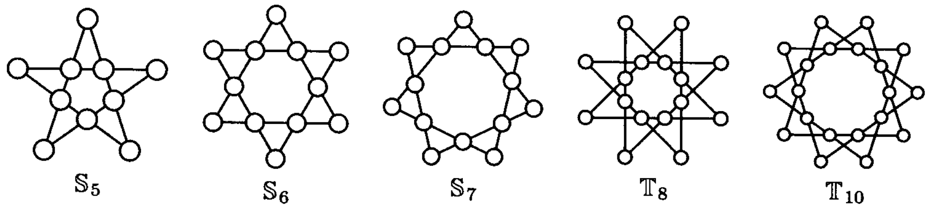


FIG. 1.

In this paper we will solve both problems and describe a generalization.

In Figure 1 five n-sided (n-vertex) stars of two types are depicted. The first three stars are denoted $\mathbb{S}_n$, $n = 5, 6, 7$, and the two others $\mathbb{T}_n$, $n = 8, 10$. Each star contains $2n$ circlets such that there are four on each line. Both types of stars arise from a regular n-gon by placing n circlets $V_1, V_2, \ldots, V_n$ centered at the vertices of the n-gon. In the star $\mathbb{S}_n$, which is defined for $n \geq 5$, the circlets $U_1, U_2, \ldots, U_n$ are situated in the intersection of lines $V_{i-2}V_{i-1}$, V_iV_{i+1}, for $i = 1, 2, 3, \ldots, n$. (subscript being taken modulo n.) In the star $\mathbb{T}_n$, $n \geq 7$, the U_i's are situated at the intersection of lines $V_{i-2}V_{i-1}$, $V_{i+1}V_{i+2}$, for $i = 1, 2, 3, \ldots, n$. See Figure 2.

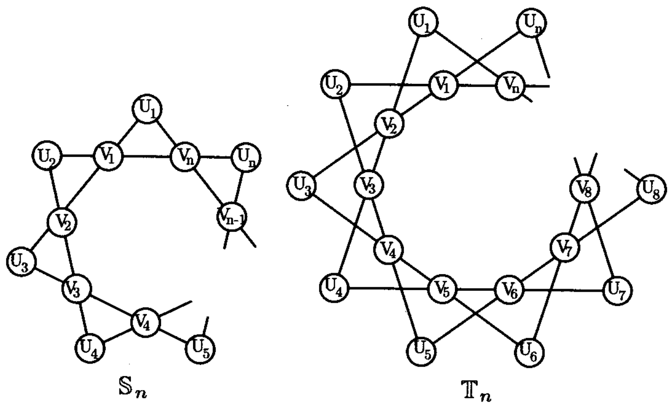


FIG. 2.

*Šafárik University

An n-sided star $\mathbb{S}_n$ (or $\mathbb{T}_n$) is called a *magic star* and labelled $\mathbb{S}_n^M$ (or $\mathbb{T}_n^M$) if the numbers $1, 2, 3, \ldots 2n$ can be placed in its circlets so that the sums of numbers on each line are the same. These stars are sometimes referred in literature as *magic stars of David.* Observe that the numbers on each line sum to twice the sum of all numbers divided by n, that is $4n + 2$.

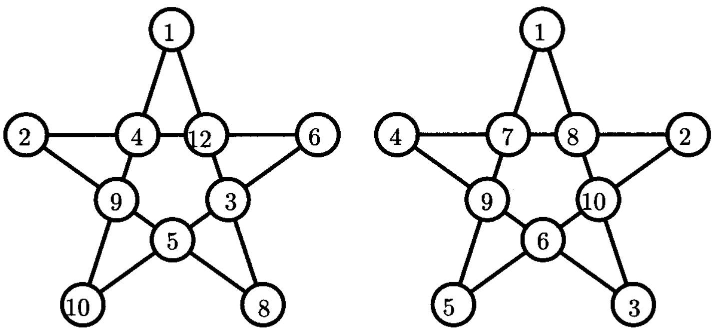
We call a star $\mathbb{S}_n$ (or $\mathbb{T}_n$) a *weakly-magic* star and denote it $\mathbb{S}_n^W$ (or $\mathbb{T}_n^W$), if distinct integers can be placed in its circlets so that the sum on each line is the same. Each magic star is also weakly-magic, however the reverse implication is not true.

If there are numbers $1, 2, \ldots 2n$ located in the circlets of a star $\mathbb{S}_n$ (or $\mathbb{T}_n$) so that the sum on $n - 2$ of the lines is $4n + 2$, and the numbers on the remaining two lines sum to $4n + 1$ and $4n + 3$ respectively, we call it an *almost magic star*, and we denote it $\mathbb{S}_n^A$ (or $\mathbb{T}_n^A$.)

Let's go back to Problem 1. Its solution is the content of the following theorem:

THEOREM 1. *The star* $\mathbb{S}_5$ *is weakly-magic and almost-magic, but it is not magic.*

Proof. In Figure 3 we see a weakly-magic star $\mathbb{S}_5^W$ and an almost-magic star $\mathbb{S}_5^A$. The sums on the lines containing the number 2 are 21 and 23 respectively.



$\mathbb{S}_5^W$ $\mathbb{S}_5^A$

FIG. 3.

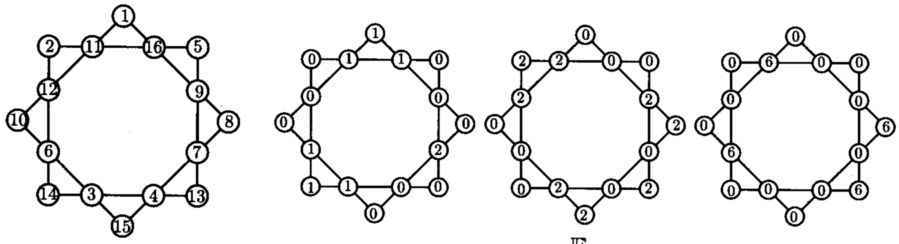
Suppose that a magic star $\mathbb{S}_5^M$ exists. On each line there are exactly four different numbers whose sum is 22. Each number is situated on exactly two lines. On individual lines there can be only the following quadruples of numbers:

$$\begin{array}{cccccc} (10,9,2,1) & (10,8,3,1) & (10,7,4,1) & (10,7,3,2) & (10,6,5,1) & (10,6,4,2) \\ (10,5,4,3) & (9,8,4,1) & (9,8,3,2) & (9,7,5,1) & (9,7,4,2) & (9,6,5,2) \\ (9,6,4,3) & (8,7,6,1) & (8,7,5,2) & (8,7,4,3) & (8,6,5,3) & (7,6,5,4) \end{array}$$

In the magic star $\mathbb{S}_5^M$ two lines must contain 10. Let's suppose that on one of these lines contains the quadruple (10,9,2,1). On the other one can be only the quadruple (10,5,4,3) as all the other quadruples containing 10 also contain 1 or 2. Similarly number 9 is also situated on two lines. Let's consider the second line containing 9. It can not contain numbers 2 and 1, therefore only the quadruple (9,6,4,3) comes into consideration, but it contains numbers 3 and 4 which are situated on the line containing number 10.

If one line contains (10,8,3,1) or (10,7,3,2) the other must contain (10,6,4,2) or (10,6,5,1), respectively. As before, we can argue that none of these options work. No other quadruples of numbers containing 10 and not containing any other common numbers do exist. □

We will now provide a solution to Problem 2. By a *basic n-sided magic star* we understand star $\mathbb{S}_n$ or $\mathbb{T}_n$, with numbers 0, k or 0, k, $2k$ written in its circlets so that the sum of the numbers on each line is the same.



$\mathbb{S}_8^M$ $\mathbb{E}_8$

FIG. 4.

Figure 4 depicts a magic star $\mathbb{S}_8^M$ and three basic 8-sided stars. We got this $\mathbb{S}_8^M$ by summing over the nine basic stars, whose valuation is shown in the rows of Table 1. The basic stars from Figure 4 are in the 3'rd and 5'th and 9'th row.

U_1	U_2	V_1	V_8	U_8	V_2	V_7	U_3	U_7	V_3	V_6	U_4	V_4	V_5	U_6	U_5
1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1
1	0	1	1	0	0	0	0	0	1	2	1	1	0	0	0
0	0	2	0	0	0	2	0	0	0	0	2	0	0	0	2
0	2	2	0	0	2	2	0	2	0	0	0	2	0	2	2
0	0	0	5	0	0	0	5	0	0	5	5	0	0	0	0
0	0	0	5	0	5	0	0	0	0	0	0	0	0	5	5
0	0	0	5	5	5	5	5	0	0	0	5	0	5	0	5
0	0	6	0	0	0	0	0	6	6	0	0	0	0	6	0
-1	0	0	0	-1	0	0	0	0	-1	0	0	0	-1	0	0
1	2	11	16	5	12	9	10	8	6	7	14	3	4	13	15

TABLE 1
Construction of $\mathbb{S}_n^M$

Figure 4 shows magic stars $\mathbb{S}_n^M$ for $n = 6, 7, 9$. Values for $\mathbb{S}_n^M$ for $n = 9, 10, 11$ are in Table 4. These were obtained by computer experimentation. The natural question for which n a magic star exists has been answered for some n, but as of yet, no general answer is known.

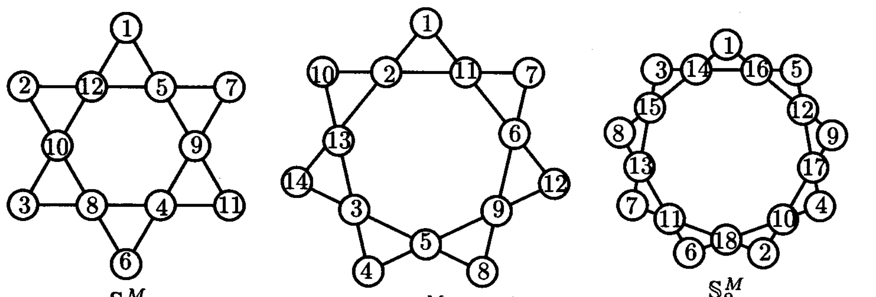
For stars $\mathbb{T}_n$, however, we do have an algorithm for producing $\mathbb{T}_n^M$ for every even integer $n \geq 8$.

THEOREM 2. *A magic star* $\mathbb{T}_n$ *exists for every even integer* $n \geq 8$.

Proof. We divide numbers $1, 2, \ldots, 2n$ into four rows as shown in Table 2.

1	2	3	4 ...	$n/2$
n	$n-1$	$n-2$	$n-3$ ...	$n/2+1$
$n+1$	$n+1$	$n+2$	$n+3$ ...	$3n/2$
$2n$	$2n-1$	$2n-2$	$2n-3$ ...	$3n/2+2$

TABLE 2



$\mathbb{S}_6^M$ $\mathbb{S}_7^M$ $\mathbb{S}_9^M$

FIG. 5.

The following relations are true:

$$i+(n-i+1)=n+1,\ (n+i)+(2n-i+1)=3n+1,\ i=1,2,\dots,\frac{n}{2},$$

$$(i+1)+(n-i+1)=n+2,\qquad (n+i)+(2n-i)=3n,\qquad i=1,2,\dots,\frac{n-2}{2},$$

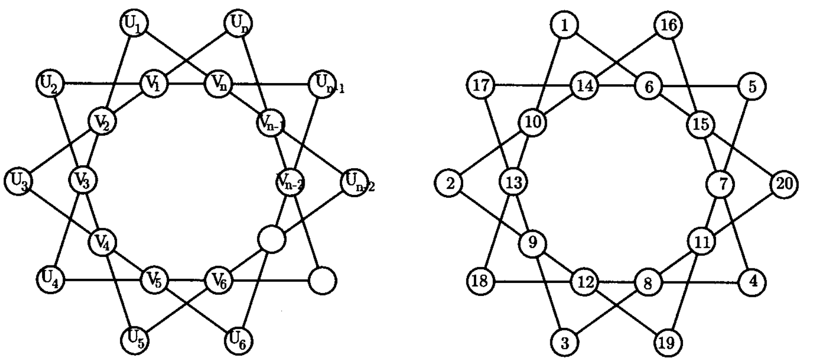


FIG. 6. *Stars* $\mathbb{T}_n$ *and* $\mathbb{T}_{10}^M$

Into the circlet corresponding to the entry in the i-th row and j-th column of Table 3 we inscribe the number from i-th row and j-th column of Table 2.

U_1	U_3	U_5	U_7	$\dots$	U_{n-3}	U_{n-1}
V_2	V_4	V_6	V_8	$\dots$	V_{n-2}	V_n
V_{n-3}	V_{n-5}	V_{n-7}	V_{n-9}	$\dots$	V_1	V_{n-1}
U_{n-2}	U_{n-4}	U_{n-6}	U_{n-8}	$\dots$	U_2	U_n

TABLE 3

On each line, except of $V_{n-1}V_n$ of the star $\mathbb{T}_n$ there are two pairs of circlets whose sums are $[(n+1)+(3n+1)]$ or $[(n+2)+3n]$. On the line $V_{n-1}V_n$ are numbers $(n/2+1), 1, 3n/2, 2n$, whose sum is $4n+2$. □

For weakly magic stars we also have a constructive existence theorem. Let U and V be two circlets of a star $\mathbb{S}_n$ (or $\mathbb{T}_n$) and let $\mathbb{E}_n$ be a basic star. We say that $\mathbb{E}_n$ *separates the circlets U and V* if it assigns to them different numbers.

THEOREM 3. *A weakly-magic star $\mathbb{S}_n^W$ (or $\mathbb{T}_n^W$) exists for every integer $n \geq 7$.*

Proof. We inscribe number 1 into each circlet of a star. We get a weakly-magic valuation of the star by multiple use of the following construction:

If the numbers in the circlets U and V are the same we choose a basic star $\mathbb{E}_n$ which separates them. By adding the valuation $\mathbb{E}_n$ to the original valuation we will get a new valuation in which different numbers will be assigned to circlets U and V. If we choose for k a number which is bigger than all the numbers in the previous valuation every two circlets which have different numbers will have different numbers also in the new valuation.

We repeat this construction as long as two different circlets with the same values exist. After a finite number of repetitions we will get a weakly-magic star. □

Note. The above shown construction leads to relatively big numbers in stars $\mathbb{S}_n^W$ and $\mathbb{T}_n^W$. If we choose basic stars and values of k appropriately we will get a magic valuation (if it exists), or as the case may be a weakly-magic star with small numbers.

	U_1	V_1	U_2	V_2	U_3	V_3	U_4	V_4	U_5	V_5	U_6	V_6	U_7	V_7	U_8	V_8	U_9	V_9	U_{10}	V_{10}	U_{11}	V_{11}
$\mathbb{S}_6^M$	1	9	5	12	4	3	6	11	8	7	2	10										
$\mathbb{S}_6^M$	29	31	53	71	73	43	37	41	47	67	59	61										
$\mathbb{S}_9^M$	1	14	3	15	8	13	7	11	6	18	2	10	4	17	9	12	5	16				
$\mathbb{S}_{10}^M$	1	3	7	20	18	4	11	14	6	8	9	16	12	15	2	10	5	17	13	19		
$\mathbb{S}_{11}^M$	1	19	7	16	10	18	5	15	3	17	9	22	4	13	2	21	8	12	11	20	6	14
$\mathbb{T}_7^A$	1	8	2	14	3	11	4	10	7	12	6	9	5	13								
$\mathbb{T}_8^M$	1	11	14	8	2	10	15	7	3	9	16	6	4	12	13	5						

TABLE 4
Some stars

Some interesting stars are contained in Table 4. In the first row a weakly-magic star $\mathbb{S}_6^M$ for which the sum of numbers in circlets U_1, U_2, ..., U_6 is the same as on individual lines. In the second row is the star S_6^W containing only prime numbers. (Such a star is in literature often referred as *prime-number magic star of David*). In the nearly-magic star $\mathbb{T}_7^A$ (6th row) the lines containing number 6 have sums different from the others.

We may ask similar questions about magic and weakly-magic valuations on different kinds of stars, for example the stars which we denote by $\mathbb{S}_7^T$ and $\mathbb{S}_9^T$, see Figure 7. In the first we placed the numbers $1, 2, \ldots, 21$ so that the sum of all six numbers on every line is sixty-six. We now pose a problem for you to solve.

Problem 3. Into the circlets of the star $\mathbb{S}_9^T$ (Fig. 7) inscribe 27 different natural numbers so that the sum of the 6 numbers on each of the lines is the same.

If you succeed in finding a magic star $\mathbb{S}_n^M$ for $n \geq 12$ or for $\mathbb{T}_n^M$ for odd $n \geq 7$, please, send me an information about it.

Finally, Figure 8 depicts a special magic star, whose properties might inspire you to formulate and investigate a wealth of similar problems.

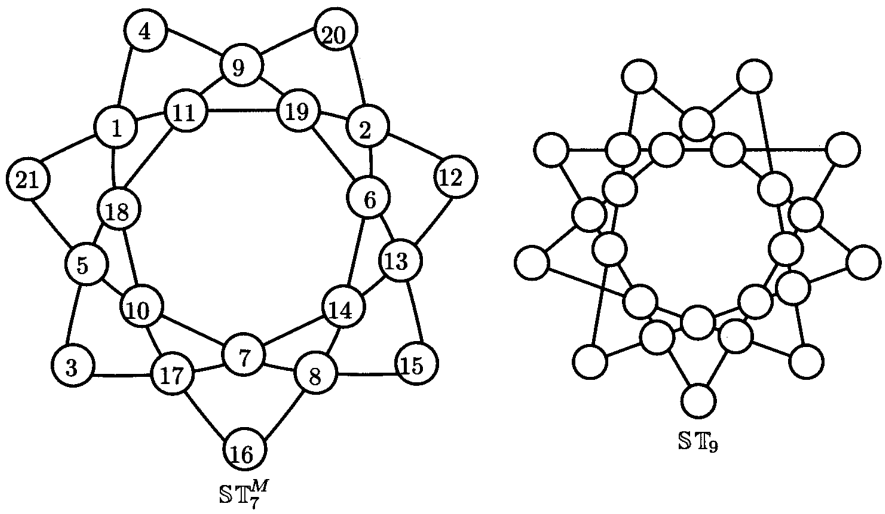


FIG. 7.

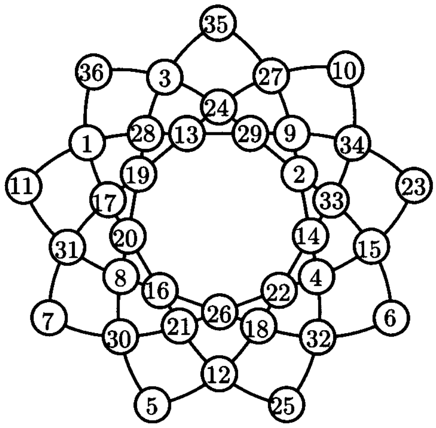


FIG. 8.

REFERENCES

[1] H. E. DUDENEY, "536 Puzzles and Curious Problems," Scribner's 1967.

Šafárik University, Jesenná 5, 041 54 Košice, Slovakia. trenkler@science.upjs.sk

Mariàn Trenkler teaches at Safarik University in Kosice, Slovakia. His areas of research are Convex Polytopes, Graph Theory, and Magic Squares and Cubes.